The dynamics of the Wealth Distribution and the Interest Rate with Credit Rationing (Thomas Piketty ReStud 1997)

Jose Ignacio Lopez

UCLA

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In the Solow Model there is unique well-defined steady state independent of the initial distribution of wealth.

When financial frictions are considered, there could be an interaction between the distribution of wealth and interest rates that causes the model to have multiple equilibria.

Long-run growth can take place in the transition from one equilibrium to another.
Basic Features of the Model

- Closed economy, discrete infinite time horizon, stationary population of unit mass.
- All agents are risk neutral
- Two goods: Labor and consumption good (which can be either consumed or invested)
- Distribution of wealth with CDF $G_t(w)$. Aggregate (average) wealth: $W_t = \int w dG_t(w)$
Households

- Households are endowed with one indivisible unit of labor and initial wealth $w_{it}$ and choose how much to consume ($c_{it}$), how much effort to incur ($e$) and how much to save ($b_{it}$)
- $U = y - e$, where $e \in \{0, 1\}$
- $c = (1 - s) y$, $w_{i, t+1} = b_{it} = sy$
- Saving rate $s$ is exogenous (as in the Solow model)
Technology has CRS and uses capital and labor $F(K, L)$

Usual Inada assumptions on $f(k)$

Stochastic shocks at the individual level:

With high effort ($e=1$)

$$f(k) = \begin{cases} f(k) \text{ with probability } p \\ 0 \text{ with probability } 1-p \end{cases}$$

With low effort ($e=0$)

$$f(k) = \begin{cases} f(k) \text{ with probability } q \\ 0 \text{ with probability } 1-q \end{cases}$$

$0 < q < p < 1$

$k^*(r)(e = 1) > k^*(r)(e = 0)$
Lemma

If \( y(0) - 1 = pf(k(r)) - (1 + r) k - 1 > y_0(0) = pf(k_0(r)) - (1 + r) k_0 \), there exists a savings rate high enough (interest rate lower than \( r^*(q) \)) that guarantees that high effort is the first best.
First Best long-run equilibrium

**Lemma**

There exist unique levels of long-run aggregate wealth \( W^* \), aggregate output \( Y^* \), interest rate \( r^* \) and inequality \( G^*(w) \) irrespective of the initial wealth distribution \( G_0(w) \)

- The marginal product of capital is equal across agents:
  \[ pf'(k) = 1 + r \]
  Optimum investment is independent of the agent’s initial wealth

\[
\begin{align*}
k(r_t) &= W_t \\
pf'(W_t) &= 1 + r \\
y_{it} &= pf(W_t) - (1 + r_t) W_t + (1 + r_t) w_{it}
\end{align*}
\]
First Best II

- Aggregate income: $Y_t (G_t) = pf (W_t)$
- $W_{t+1} = sY_t = spf (W_t)$
- $W^* = spf (W^*)$
- Transitional equations: (risk-neutrality is important here);

$$w_{it+1} (w_{it}) = \begin{cases} s \left[ f \left( k (r) + (1 + r) \frac{(w - k(r))}{p} \right) \right] & \text{with probability } p \\ 0 & \text{with probability } 1-p \end{cases}$$
Credit - Constrained I

- Effort is unobservable, so lenders must check beforehand whether the borrowers have incentives to shirk.
- After the contract is signed (ex-post), agents will incur in high effort only if

\[ p \left[ f \left( k (r) - d_s \right) \right] - 1 > q \left[ f \left( k (r) - d_s \right) \right] \]

- The optimal contract specifies repayments depending on whether the project fails (\(d_f\)) or succeeds (\(d_s\)).
Lenders should recover in expected term their investment and receive the interests.

\[
\begin{align*}
    d_f & = 0 \\
    d_s & = \frac{(1 + r)(k(r) - w)}{p} \\
    pd_s + (1 - p)d_f & = (1 + r)(k(r) - w)
\end{align*}
\]
Incentive Constraint:

\[ p [ f ( k ( r ) - d_s )] - 1 > q [ f ( k ( r ) - d_s )] \]

- If \( w < w ( r ) = k ( r ) - \left[ pf ( k ( r )) - \frac{p}{p-q} \right]_{1+r} \) the IC is not satisfied.
  All agents will wealth below this threshold level will supply the minimal effort.
Mapping from q to r(q)

Lemma

There exists $q_0 > 0$ ($q_0 < p$), such that for any $q \in (0, q_0)$, there exists $r(q) \in (0, r^*(q))$ such that

1. If $r \leq r(q)$ there is no credit rationing: all households achieve the first-best investment level

2. If $r(q) \leq r \leq r^*(q)$, there are some agents credit constrained: $\exists w(r) > 0$, such that if $w_i < w(r) \rightarrow e = 0, k = k_o$

3. If $r > r^*(q)$, all agents choose to make the low investment
Dynamics

- If \( r > r(q) \), the transition functions for agents whose \( w_{it} < w(r) \) is:

\[
w_{it+1}(w_{it}) = \begin{cases} 
  s \left[ f(k_0(r)) - \frac{(1+r)(k_0(r)-w)}{q} \right] & \text{with probability } q \\
  0 & \text{with probability } 1-q 
\end{cases}
\]

- The distribution of wealth is relevant to determine allocations:

\[
W_t = G_t(w(r_t)) k_0(r_t) + [1 - G_t(w(r_t))] k(r_t)
\]
Low fraction of steady state constrained agents

Figure 3
Individual transitions with credit-rationing ($r(q) < r^*$)
High fraction of steady state constrained agents

\[ sy(r')/q \]

\[ W_{n+1} \]

\[ W_n \]

**Figure 4**

Individual transitions with credit-rationing \([r(q) < r < r' < r^*(q)]\)
Credit-constrained agents supply less capital than unconstrained agents \((spf(k(r)) > sqf(k_0(r)))\) but also demand less capital \(k_0(r) > k(r)\).

If the first effect dominates, a higher fraction of constrained people will push interest rates up. (this requires:
\[
\frac{pf(k(r))}{k(r)} > \frac{qf(k_0(r))}{k_0(r)} \rightarrow \frac{f(k(r))}{kf'(k(r))} \text{ decreasing function of } k \rightarrow \text{For a CES} \quad \frac{pf(k(r))}{k(r)} = 1 + k^{\frac{1-\sigma}{\sigma}}, \sigma < 1
\]

Starting from an equilibrium with high-accumulation and low interest rates, a positive shock on the interest rate can be self sustaining if it pushes many agents into the credit rationing region, so savings are lower and interest rates are higher.
Lemma

To each possible stationary interest-rate $r_\infty \in [0, r^* (q)]$ corresponds a unique stationary, ergodic distribution $G_\infty (w)$. Then $r_\infty$ is a long-run steady-state interest rate of the dynamic system $(G_{t+1} (G_t), r_t = r (G_t))$ iff $r_\infty = r (G_{r_\infty})$. 
There can be long-run growth in this model if shocks to the interest rate are self-sustained.

Arguably, the conclusions are not robust if one drops the assumption that the saving rate is constant and agents are risk-neutral (lack of insurance motive).

Question: Can the multiplicity disappear if agents face stochastic interest rates shocks and agents cannot always adjust their savings?
\[ w(r) = k(r) - \frac{1}{1+r} \left[ pf(k(r)) - \frac{p}{p-q} \right]. \]

If \( w(r) < 0 \) (so all agents are unconstrained), \( k(r) < \frac{1}{1+r} \left[ pf(k(r)) - \frac{p}{p-q} \right] \). At \( r = 0 \),

\[ \frac{p}{p-q} < pf(k(0)) - k(0). \]

Note that

\[ w'(r) = k'(r) - \left[ pf(k'(r))k'(r)(1+r) - pf(k(r)) \right] \left( \frac{1}{1+r} \right)^2 - \frac{p}{p-q} \frac{1}{(1+r)^2} = \]

\[ \frac{1}{(1+r)^2} \left[ pf(k(r)) - \frac{p}{p-q} \right] \]

\( y_0(r) \) reaches its maximum at \( k_0 \). Then,

\[ pf(k(r)) - (1+r)k(r) - qf(k(r)) - (1+r)k(r) > y(r) - y_0(r) > 1 \]

for \( r < r^*(q) \rightarrow pf(k(r)) - \frac{p}{p-q} > 0 \rightarrow w'(r) > 0. \]